Inflation and asymptotic safety

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Abstract

The paper presents the key idea behind Asymptotic Safety in gravity and its application to the inflationary models. It is discussed, that AS inflation is possible, by the RG improvement of the Starobinsky model at the level of the Lagrangian. The Higgs inflation and AS gauge-Yukawa theories are addressed in this context.

1 Introduction

Cosmological inflation is one of the most established theories, describing the early Universe. Inflation has brought answers to numerous questions in classical cosmology, such as observed homogeneity and isotropy of the Universe. Remarkably, quantum fluctuations present at the beginning of the inflation have been enhanced to a classical level and are directly observed as Cosmic Microwave Background radiation, so far the theory is consistent with the data. However, the modern approach to inflation creates many possible models, of which one (if any) is correct. Novel theoretical tools are necessary to narrow down the possible scenarios.
In the paper, we review inflationary models through the lens of Asymptotic safety, which as noticed by Steven Weinberg [4] gives a natural rise for the inflation, by including all possible truncations of Riemann curvature tensor in the action.
The paper is organized as follows, in Sec. 2 a brief comment on the relevance of the inflation is given, with the references to the literature. Sec. 3 introduces the main aspects of Asymptotic safety, Renormalization Group flow, and fixed points. Sec. 4 consists of two appealing models, currently with an agreement with the experimental data. Namely, Starobinsky inflation, in which the inflationary potential originates from a quadratic term in the action and Higgs particle inflation, which considers the Higgs particle as the Inflaton. Sec. 5 in

*In order corresponding to the section written by the author.

detail describes RG flow in the gauge-Yukawa theory with $\mathrm{SU}(N_c)$ gauge fields. The stability of the found fixed points is discussed. In Sec. 6 Weinberg's original idea is explained, as well as the modern realization of Alessia Platania. In Appendix A Jordan and Einstein frame has been introduced. Appendix B consists of the explicit form of the potential of previously described g-Y Asymptotically Safe theory.

2 Inflation theory

Current observations can reach no further in the past than to the surface of the last scattering where the cosmic microwave background comes from. The relic radiation is an extremely useful source of information about the era of recombination as well as a tool discarding some of the early universe models.

The standard big-bang cosmologies' conditions miss some of the observational features of the universe. The most meaningful ones are the flatness, homogeneity, and primordial monopole problems [14] [13]. Under a certain initial condition, the inflation theory is the one enabling avoidance of this inconsistency.

Since the 80s [14] the theory has been modified and tested multiple times. The inflation mechanism is caused by the scalar potential field, leading to particle productions and exponential expansion of the universe as the initial false vacuum gives up its energy, cooling the space. The potential rolls down, creating the particles.

Some of the models predict a bouncing universe and some of the initial conditions may lead to eternal inflation. In this case, the expansion occurs in various regions of the universe differently, which can be troublesome as we don't observe any evidence of this process taking place.

3 Asymptotic safety

The attempts to quantize general relativity based on Einstein-Hilbert action result in perturbatively nonrenormalizable theory. This prompted researchers to treat gravity as an effective field theory. In this scheme, the predictive power of the theory is limited because the description of gravity at trans-Planckian scales requires fixing infinitely many coupling constants from experiments [1]. The idea of asymptotic safety was introduced by Stephen Weinberg in 1978 as a UV completion of the quantum theory of gravity. The behavior of an asymptotically safe theory is characterized by scale invariance in the high-momentum regime. The realization of scale invariance requires the existence of a nontrivial renormalization group fixed point for dimensionless couplings. Dimensionless couplings g_i are obtained from dimensionful ones with canonical dimension $d_{\bar{g}_i}$ by multiplication by suitable power of the renormalization group scale k:

$$g_i(k) = \bar{g}_i(k) k^{-d_{\bar{g}_i}}.$$

The fixed point of the theory g_* is a zero of all beta functions for couplings g_i:

$$\beta_{g_i} = k\partial_k g_i(k) = 0 \text{ for } g_i(k) = g_{i*}.$$

The trajectories of the RG flow can be visualized in the theory space. It is spanned by field monomials of the theory; each point of the space is associated with a possible action, which is a linear combination of the field monomials. The existence of the RG fixed point results in universal predictions for low-energy physics. It determines relations between couplings and thus describes the location of the UV-critical surface. The UV-critical surface consists of all couplings along which the trajectories emanate from the fixed point in the infrared (IR) direction with a diminishing scale k. These couplings correspond to relevant directions; they are UV-attractive as they reach the fixed point at a high-energy scale. The relevant directions are the free parameters of the theory. The IR-attractive couplings (which are UV-repulsive) constitute irrelevant directions as they are pulled automatically towards the fixed point [2]. The dimensionality of the UV-critical surface is equal to the number of relevant directions. Figure 1 presents the fixed point with an associated UV-critical surface in the theory space.

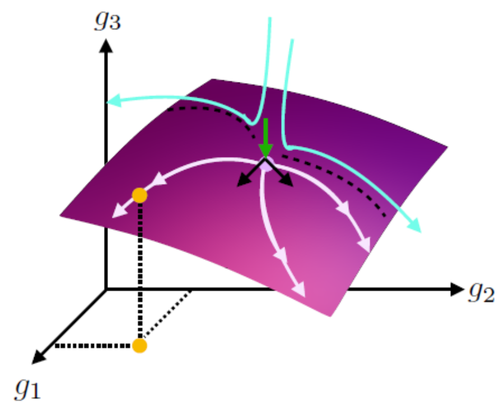

Figure 1: Fixed point (light purple dot) with corresponding UV-critical surface (purple) in the theory space. Teal arrows show RG trajectories starting off the surface pulled toward the fixed point along the irrelevant direction until the relevant (IR-repulsive) directions start to drive the flow away from the fixed point. Linearized flow is visualized by black (relevant directions) and green (irrelevant directions) arrows [2].

Asymptotic safety imposes the condition that the theory describing nature lies on the UV-critical surface of the theory space. This condition implies that the couplings of the theory are finite at high-energy. Furthermore, to fix the trajectory uniquely one has to determine free parameters from experiments[1].

The number of parameters equals the dimensionality of the UV-critical surface; the finite dimensionality of the critical surface results in a finite number of experiments needed and restores the predictivity of the theory. The difference between asymptotic safety and asymptotic freedom present in e.g. QCD is that asymptotic freedom manifests in the asymptotic weakening of the coupling in the high-energy limit. Asymptotic safety, besides stabilizing behavior of the theory at high energy by scale invariance, generates predictions in low-energy physics by relations between relevant and irrelevant directions.

In order to determine the UV-attractive directions one has to investigate the linearized flow about the fixed point at $g = g_*$:

$$\beta_{g_i} = \sum_j \frac{\partial \beta_{g_i}}{\partial g_j}\Big|_{g=g_*} (g_j - g_{j*}) + \mathcal{O}(g_j - g_{j*})^2.$$

The solution of the equation above can be written in the form:

$$g_i(k) = g_{i*} + \sum_I c_I V_i^I \left(\frac{k}{k_0}\right)^{-\theta_I},$$

where: c_I-constants of integration, θ_I are the critical exponents; they are related to eigenvalues of stability matrix $\mathcal{M}_{ij}$:

$$\theta_I = -\mathrm{eig}\mathcal{M}_{ij} = -\mathrm{eig}\frac{\partial \beta_{g_i}}{\partial g_j}\Big|_{g=g_*}$$

while V_I are corresponding eigenvectors[2].

One of the powerful methods for examining RG flow in the context of asymptotic safety is the functional renormalization group (FRG) study for gravitational effective average action (EAA) Γ_k. The EEA scale dependence is described by the Wetterich equation (also called the FRG equation). In practical applications, the theory space is often truncated. For the Einstein-Hilbert truncation, the effective average action contains scalar curvature, the cosmological constant, gauge fixing, and gauge ghost term. It has been shown that there exist Gaussian (asymptotically free theory) and a non-Gaussian fixed point in such a theory. Figure 2 shows the corresponding RG flow.

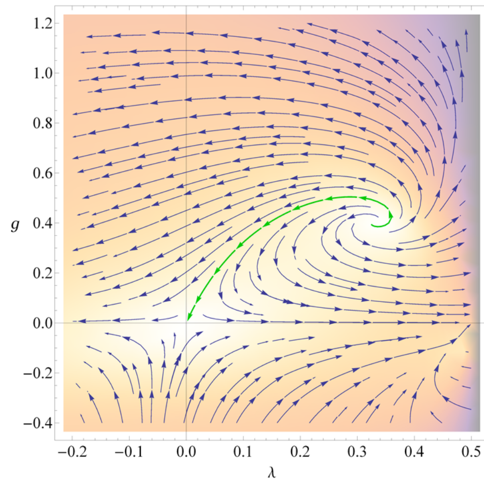

Figure 2: Visualization of the RG Einstein-Hilbert flow. The plot shows Gaussian fixed point in $g = 0$, $\lambda = 0$ and a non-Gaussian fixed point.

Numerous quantum theories of gravitation, such as R^2 gravity, Weyl tensor squared gravity, and $f(R)$ gravity has been confirmed to contain the non-Gaussian fixed point.

4 (Semi)successful inflationary models

4.1 Starobinsky inflation

In 1980 Starobisky [7] proposed a model where a pure modified gravitational action can cause non-singular evolution of the Universe, namely:

$$S = \frac{1}{2}\int \sqrt{|g|}d^4x \left(M_p^2 R + \frac{1}{6M^2}R^2\right),$$

where M is some "mass" parameter, with value taken to fit the Planck data. Now we will rewrite the action into equivalent linear representation:

$$S_l = \frac{1}{2} = \frac{1}{2}\int \sqrt{|g|}d^4x \left(\frac{M_p^2}{2}R + \frac{1}{M}R\psi - 3\psi^2\right),$$

if we write equations of motion for ψ we obtain:

$$\frac{1}{M}R = 6\psi.$$

Then if we use a following conformal transformation:

$$g_{\mu\nu} \to e^{-\sqrt{2/3}\phi/M_p} g_{\mu\nu} = \left(1 + \frac{2\psi}{MM_p^2}\right) g_{\mu\nu}$$

we get action with scalar field coupled to gravity:

$$S = \frac{1}{2}\int d^4x\sqrt{|g|}\left(\frac{M_p^2}{2}R + \frac{1}{2}\partial_\mu\partial^\mu\phi - \frac{3}{4}M_p^4 M^2\left(1 - e^{-\sqrt{2/3}\phi/M_p}\right)\right).$$

R^2 Term gives equivalent solutions as the evolution of scalar field with exponential type potential. According to Planck data, Starobinsky model and its descendants are the main class of models which has correct tensor to scalar ratio and scalar-tilt:

$$n_s - 1 \approx -\frac{2}{N} \qquad\qquad r \approx \frac{12}{N^2},$$

with N being the number of e-folds.

4.2 Higgs particle as Inflaton

The inflation scenario requires a scalar field to drive it. We can assume a fictitious, additional to the ones we know from particle physics, scalar field to do the job, but this will cause a lot of problems: how to quantize this field, how will it couple to other fields from SM (reheating), what other properties should it have. So far we recognized only one fundamental scalar field, namely the Higgs field. Then arises a question, whether Higgs can serve as Inflaton. The answer is yes, moreover the spectral index and tensor perturbations amplitude for SM are in good agreement with the experiment and these parameters are in 15 correspondence to WMAP-3 data. In this paragraph, we will follow the steps described in [8].
Let us start with Standard Model Lagrangian with non-minimal coupling to gravity:

$$\mathcal{L} = \mathcal{L}_{SM} - \frac{M^2}{2}R - \xi H^\dagger H R$$

We will consider only ξ such that: $1 \lll \sqrt{\xi} \lll 10^{17}$ since this will simplify formulas and in which: $M \simeq M_P$. Let us ignore gauge couplings and set unitary gauge: $H = h/\sqrt{2}e^{i\theta}$

$$S_H = \int d^4x\sqrt{|g|}\left[-R + \frac{\partial_\mu h\partial^\mu h}{2} + \frac{h^2\partial_\mu\theta\partial^\mu\theta}{2} \; - \frac{\lambda}{4}(h^2 - v^2)^2\right]$$

We will change the frame. To obtain this we will use a conformal factor:

$$\Omega^2 = 1 + \frac{\xi h^2}{M_P^2}$$

so the transformed metric is:

$$g_{E\mu\nu} = \Omega^2 g_{J\mu\nu}$$

Moreover, if we use a convenient new scalar field such that

$$\frac{d\chi}{dh} = \sqrt{\frac{\Omega^2 + 6\xi^2 h^2/M_P^2}{\Omega^4}} \tag{1}$$

We arrive at the action in the Einstein frame:

$$S = \int d^4x\sqrt{g_E}\left[-\frac{M_P^2}{2}R_E + \frac{\partial_\mu\chi\partial^\mu\chi}{2} - U(\chi)\right],$$

where

$$U(\chi) = \frac{1}{\Omega^4}\frac{\lambda}{4}(h(\chi)^2 - v^2)^2$$

For small field values: $h \simeq \chi$ and $\Omega^2 \simeq 1$ for both fields potential has the same initial values. However, it is not so for $h \gg M_P/\sqrt{\xi}$. In this limit one can solve (1) and get:

$$h \simeq \frac{M_P}{\sqrt{\xi}}\exp\left(\frac{\chi}{\sqrt{6}M_P}\right)$$

We obtain exponentially flat potential:

$$U(\chi) = \frac{\lambda M_P^4}{\sqrt{\xi}}\left(1 + \exp\left(-\frac{2\chi}{\sqrt{6}M_P}\right)\right)^{-2}$$

We will analyse this potential using slow - roll approximation. We can calculate slow-roll parameters, in the limit $h^2 \gg M_P^2/xi \gg v^2$ as:

$$\epsilon = \frac{M_P^2}{2}\left(\frac{dU/d\chi}{U}\right) \simeq \frac{4M_P^4}{3\xi^2h^4},$$

$$\eta = M_P^2\left(\frac{d^2U/d^2\chi}{U}\right) \simeq \frac{4M_P^4}{3\xi h^2},$$

Slow roll ends when $\epsilon \simeq 1$, so $h_{end} \simeq 1.07M_P/\sqrt{\xi}$. The number of e-foldings is given by the formula:

$$N = \int \frac{1}{M_P^2}\frac{U}{dU/dh}\left(\frac{d\chi}{dh}\right)^2 dh \simeq \frac{6}{8}\frac{h^2}{M_P^2/\xi}$$

For all values $\sqrt{\xi} \ll 10^{17}$, the v parameter doesn't appear anywhere so inflation stage is not affected by its value. one obtains a familiar relation:

$$n \simeq 1 - 2\eta \simeq 1 - 2/N \simeq 0.97$$

$$r = 16\epsilon \simeq 12/N^2 \simeq 0.0033,$$

where $N \simeq 60$ Inserting into COBE normalisation [9]: $U/\epsilon = (0.027 M_P)^4$ and with $N_{COBE} \simeq 62$, we obtain that:

$$\xi \simeq \sqrt{\frac{\lambda}{3}} \frac{N_{COBE}}{0.027^2} \simeq 49000\sqrt{\lambda},$$

so for $\lambda \sim O(1)$, $\xi \simeq 49000$ is the value for which Higgs scenario fit the data. One have to keep in mind that both those models do not have UV completion and have issues with unitarity ($\xi \approx 10^5$). Now we will try to explore alternative solutions which indeed have UV completion and lead to smaller ξ. We will focus on asymptotically safe theories, which guaranty that such models are proper on every scale. They are as well a conformal field theory around fixed point, so it gives us a natural image of almost invariant spectrum in respect to scaling ($n_s \approx 1$).

5 Inflation from Asymptotically Safe Theories

5.1 Motivation

It seems very natural for a theory underlying the inflation to be fundamentally well-defined, including arbitrary short scales. Fortunately, such a nontrivial example was proposed in [15, 16]. Here we briefly summarize the content of them, describe the introduced model and its properties. Next, following [17], we consider it as a mechanism driving the inflation and compare it with physical parameters measured via CMB observations [18].

5.2 Asymptotic safety guaranteed

The UV fixed points (FP) are central for QFT theories to be fundamental and predictive up to the highest energies. Even not Asymptotically Free (AF) or power-counting renormalizable theories may turn out to be predictive, provided that they develop an interacting UV FP. Asymptotic safety (AS) guarantees UV finite matter-gauge theories even when AF or supersymmetry is not present. The idea is that AS UV FP should act as an anchor for the RG evolution of couplings so that they approach high-energy limit along well-defined RG trajectories (without divergencies such as Landau poles). Examples of UV FPs arising for gravitons, fermions, gluons, and scalar fields are given. The paper deals with the so-called Gauge-Yukawa (g-Y) theory with SU(N_C) gauge fields, N_F flavors of fermions and $N_F \times N_F$ complex matrix scalar field H (uncharged under the gauge). The Lagrangian $\mathcal{L}$ reads

$$\begin{aligned}\mathcal{L} =& \mathcal{L}_{YM} + \mathcal{L}_F + \mathcal{L}_Y + \mathcal{L}_H + \mathcal{L}_U + \mathcal{L}_V \\ =& -\frac{1}{2}\mathrm{Tr}F^{\mu\nu}F_{\mu\nu} + \mathrm{Tr}(\overline{Q} i \not{D} Q) + y\mathrm{Tr}(\overline{Q}_L H Q_R + \overline{Q}_R H^\dagger Q_L) + \mathrm{Tr}(\partial_\mu H^\dagger \partial^\mu H) \\ & - u\mathrm{Tr}(H^\dagger H)^2 - v(\mathrm{Tr}H^\dagger H)^2\end{aligned}$$

where

$$Q = Q_L + Q_R,\ Q_{L/R} = \frac{1}{2}(1 \pm \gamma_5)Q.$$

The trace is taken over color and flavor. The ratio $\delta = N_F/N_C$-11/2 plays the role of a perturbative control parameter and is later used in power series expansions. The analysis is performed in the large-N (Veneziano) limit, however, some remarks about finite N setting are given at the end of the paper. The RG flow concerns four normalized couplings α_i, where $i = g, y, h, v$, of the couplings g, y, u, v respectively. The analysis up to the leading order (LO), next-to-leading order (NLO) and next-to-next-to-leading order (NNLO) is performed. Within the analysis, the authors look at the RG flows and identify fixed points, with their types. The eigenvalues and eigenvectors of the stability matrix corresponding to the linearisation of RG flow in the vicinity of the UV fixed point are given. The theory develops a Gaussian FP which remains at zero in all orders. The dynamics of α_v largely decouples from the remaining couplings. The theory develops AS UV FP in the g-Y system at the NLO level, which bifurcates into several UV FPs at NNLO level, due to scalar fluctuations. In total, there are three non-Gaussian FPs (FP_1, FP_2, FP_3), first two of which are completely (i.e. in all four couplings) AS, and in the third point the β-function of the doubletrace scalar coupling does not vanish. The separation of eigenvalues to relevant and irrelevant in all three points is given. To sum up, the theory is renormalizable within the perturbative theory (PT). It becomes AF in gauge sector for $\delta < 0$ and QED-like for $\delta > 0$. It develops an exact interacting UV FP which is strictly controlled by the PT for $0 < \delta \ll 1$.

Then, the authors look at the UV critical surface, which describes the short-distance behavior of the theory. The analysis of anomalous dimensions and mass-terms is given. Finally, there is a discussion of the feasibility of results and additional observations using different arguments. First of them is stability which manifests itself in the fact that the leading coefficients of FPs α_g^* and α_y^* remain numerically unchanged when passing from NLO to NNLO. This also holds for universal eigenvalues. Moreover, all couplings of the theory become fully dynamic at NNLO. Crucially, at N^3LO there are no new consistency conditions, only higher-order corrections. The second argument concerns the Weyl consistency conditions. Third - the universality, means that an interacting UV FP arises universally, i.e. irrespectively of the regularisation scheme. This can be seen from the fact that FP in the gauge sector is invariant to LO under perturbative (non-singular) reparametrizations. Next, we have operator ordering and the existence of a gap Δ (of the eigenvalue spectrum). Note that the canonical power counting, unlike in AF theories, cannot be used to determine which invariants will become relevant. The residual interactions, even if perturbatively weak, control the scaling of invariants which classically have a vanishing canonical mass dimension and can change these into relevant or irrelevant ones Classically we have the fourfold degeneracy of the marginal invariants and $\Delta = 0$. Residual interactions at the UV FP lift the fourfold degeneracy amongst the classically marginal couplings and the gap arises. Net we have the unitarity condition - an important constraint on quantum corrections that relates to the scaling dimension of primary fields such as scalar fields themselves. For a quantum theory to be compatible with unitarity, it is required that the scaling dimension must be larger than unity, i.e. $\Delta_H > 1$. This behavior can be observed in the paper's result. Finally, the (lack of) triviality is discussed. It turns out that at UV FPs triviality for all three types of fields is evaded through residual interactions. This suggests that the scalar degrees of freedom may indeed be seen as elementary. The avoidance of triviality in the scalar sector is closely linked to the presence of gauge fields, be they AF or AS. It is worth noting that an interacting FP in the scalar sector would not arise without an interacting FP for the Yukawa coupling. Without gauge fields, the fermion-boson subsystem does not generate an interacting UV FP, and couplings cannot reach the GFP in the UV. With AF GFs (i.e. for small $\epsilon < 0$), the UV FP for the Yukawa coupling remains the trivial one. A detailed inspection then shows that complete AF follows, albeit under certain constraints on the parameters. With AS GFs (i.e. for small $\epsilon > 0$), complete AS is achieved at two interacting UV FPs. Triviality is evaded in the large-N limit with and without AF in the gauge sector. AF in the gauge sector had to be given up for the Yukawa and scalar sectors to develop interacting UV FPs.

5.3 Vacuum stability of asymptotically safe gauge-Yukawa theories

In the first paper, during the analysis of the RG flow, it was assumed that the vacuum of the scalar potential stays at the origin, such that all global symmetries are preserved along the flow. The main goal of the second paper is to justify this assumption.

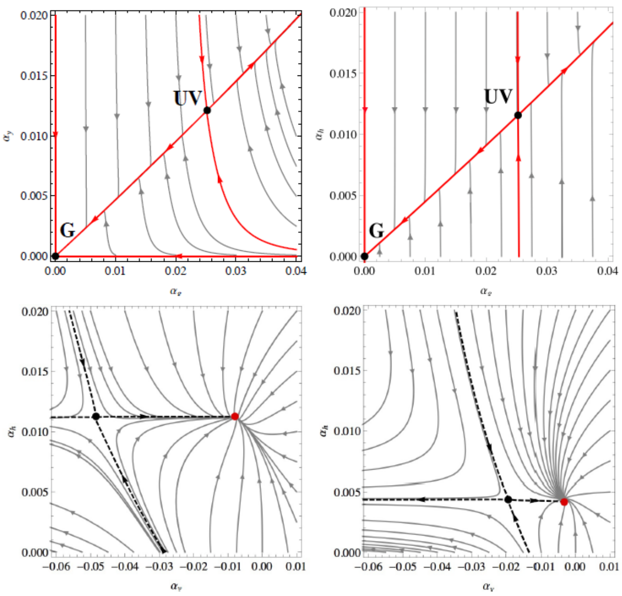

Figure 3: UP: the phase diagram of the g-Y theory in the vicinity of the UV FP at NNLO accuracy with $\epsilon = 0.05$, projected onto the (α_g, α_y) plane (left panel) and the (α_g, α_h) plane (right panel) Down: projection of the phase diagram of the g-Y theory onto the subspace of scalar couplings with (α_g, α_y) taking values on the UV-IR connecting separatrix (left panel: $\alpha_g \approx 0.999\alpha_g^*$, right panel: $\alpha_g \approx 0.397\alpha_g^*$). Red (black) dots indicate the trajectory which connect the physical FP1 (unphysical FP2) with the GFP. In the scalar subsystem where the RG flow is parametrically faster by $1/\delta$, the separatrices appear as pseudo FPs.

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The analysis concentrates on the separatrix which connects UV FP with GFP. In coincides with UV critical surface close to FP. It is characterized by locations of FPs and eigendirections. Out of four eigenvalues at the FP, three are irrelevant (positive) and one is relevant (negative). The irrelevant eigenvalues are of order δ and the relevant one is of order δ^2. Thus, the velocity of RG flow along the separatrix is $\propto \delta^2$, and towards it is $\propto \delta$. Hence, for small δ, the flow goes effectively along the separatrix. The relations between couplings on the separatrix are calculated and the effective RG running of α_g along the separatrices are found. Then, there is a discussion about the characteristic energy scales and, so-called, dimensional transmutation. The figure of RG running of all couplings along the UV-IR connecting separatrix in the NLO approximation is given. It depicts the cross-over (in IR regime the couplings converge are close to each other, i.e. to zero; in UV regime they stabilize at different (non-zero) levels) which takes place at RG-invariant scale. Then the stability analysis begins. It involves classical and quantum moduli spaces. The classical moduli space for the potential

$$V = u\mathrm{Tr}(H^\dagger H)^2 + v(\mathrm{Tr}H^\dagger H)^2$$

is just the set of flat directions. The analysis shows that FP_1 is stable, FP_2 is unstable and FP_3 is unstable for infinite N_F and possibly stable for finite N_F. Later, the authors perform the analysis of quantum moduli space. This involves e.g. the notion of the Coleman-Weinberg's potential. An enhancement of the quantum effective potential over the classical one is presented. The paper concludes that the high-energy behavior is AS, controlled by an exact interacting UV FP. RG trajectories emanating from the fixed point relate to well-defined, finite, and predictive local QFTs at all energies, despite no AF present. The FP occurs parametrically close to the Gaussian and admits rigorous control within PT The vacuum of UV safe g-Y theories is stable, (classically and QM), even though AF is absent. The main quantum corrections to V_{eff} arise due to the anomalous dimension of the scalars (non-vanishing value even at the highest energies, unlike in AF theories). The RG running of couplings away from the FP is a subleading effect for the V_{eff}, provided field values remain large compared to Λ_c (i.e. the characteristic energy scale) of the theory. The absence of classically flat directions of the FP potential thus entails quantum stability. Proof of vacuum stability can straightforwardly be exported to other gauge theories with interacting UV FPs. They investigated massless theories in the Veneziano limit where the number of fields is very large and UV interactions are weak. Continuity in the number of fields indicates that the vacuum remains stable even for finitely many fields as long as PT remains a good approximation.

5.4 Gravity and inflation

Asymptotically, potential of such an AS theory, derived in appendix B.1, can be expressed as:

$$\lim_{\phi/\mu_0 \to \infty} V_{\mathrm{iUVFP}} = \frac{\lambda_* \phi^4}{4N_F^2} \left(\frac{\phi}{\mu_0}\right)^{-\frac{16}{19}\delta} .$$

Using $\delta > 0$ we can control the hight of the potential and thus the amplitude of scalar perturbations, overall coupling depends only on the number of colors and flavors. From now we assume $\mu_0 = 10^{-3} M_P$. In our case, action in Jordan frame is given by:

$$S_J = \int d^4x \sqrt{-g} \left(-\frac{M^2 + \xi\phi^2}{M_P^2} R + \frac{g^{\mu\nu}}{2} \partial_\mu\phi \partial_\nu\phi - V_{\mathrm{iUVFP}} \right),$$

where ξ is coupling constant in general case, including non-minimal coupling. Now we can transform our action to Einstein frame as explained in the appendix A:

$$\Omega^2 = \frac{M^2 + \xi\phi^2}{M_P^2}, \quad U = V_{\mathrm{iUVFP}}/\Omega^4, \quad \phi \to \chi.$$

From now we will assume $M = M_P$. Let us recall canonical definition of slow-roll parameters and the end-of-inflation conditions:

$$\epsilon = \frac{M_P^2}{2}\left(\frac{dU/d\chi}{U}\right)^2, \ \epsilon(\phi_{end}) = 1,$$
$$\eta = M_P^2 \frac{d^2U/d\chi^2}{U}, \ |\eta(\phi_{end})| = 1,$$
$$N = \frac{1}{M_P^2}\int_{\chi_{end}}^{\chi_{ini}} \frac{U}{dU/d\chi} d\chi = 60 \text{ - number of e-folds.}$$

Scalar perturbation parameters estimated eg. in Planck'15 [18] can be expressed using slow-roll parameters in a following way:

$$\begin{aligned} A_s =& \frac{U}{24\pi^2 M_P^4 \epsilon} - \text{amplitude}, \\ n_s =& 1 + 2\eta - 6\epsilon - \text{tilt}, \\ r =& 16\epsilon - \text{tensor-to-scalar ratio.} \end{aligned}$$

5.5 Minimal coupling

At first let us assume $\xi = 0$, so called minimal coupling case.

$$\begin{aligned} \phi_{ini} =& \sqrt{\left(4 - \frac{16}{19}\delta\right)\left(3 - \frac{16}{19}\delta\right)} M_P \\ \phi_{end} =& \sqrt{\left(4 - \frac{16}{19}\delta\right)\left(2N + 3 - \frac{16}{19}\delta\right)} M_P \\ n_s =& \frac{2N - 3}{2N + 3 - \frac{16}{19}\delta} = 0.951 + 0.00651\delta + \mathcal{O}(\delta^2) \\ r =& \frac{32\left(1 - \frac{4}{19}\delta\right)}{2N + 3 - \frac{16}{19}\delta} = 0.260 - 0.0530\delta + \mathcal{O}(\delta^2) \end{aligned}$$

After comparison of our results with Planck experiment (Fig.4), we can clearly see, that our parameter δ has to be around 0.8 to fit in reality. Although this is still in theory's perturbational region, we have to consider higher order terms (B.2 Fig.9).

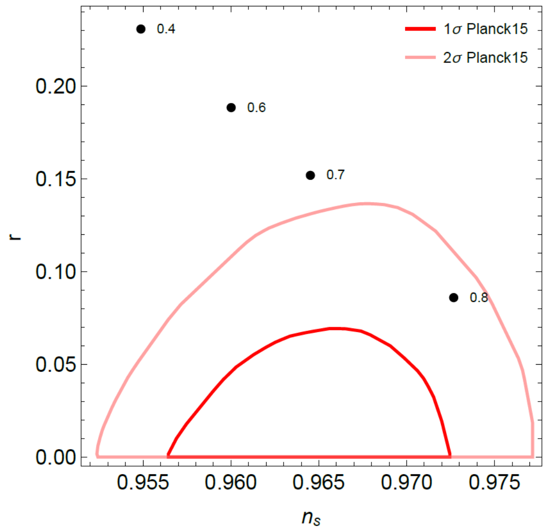

Figure 4: Theoretical predictions for different δ values computed with complete expression for potential [17][18].

We can now use the last parameter A_s to determine exact number of theory's flavours and colours (B.2 Fig.10).

$$
\begin{aligned}
A_s = & \frac{\lambda^*}{48\pi^2\left(4-\frac{16}{19}\delta\right)^2 N_F^2}\left(\frac{\phi_{ini}}{M_P}\right)^{6-\frac{16}{19}\delta}\left(\frac{\mu_0}{M_P}\right)^{\frac{16}{19}\delta} \\
& \approx \frac{10^5\delta}{N_F^2}\left(\frac{\mu_0}{M_P}\right)^{\frac{16}{19}\delta} = 2.2\cdot 10^{-9}
\end{aligned}
$$

5.6 Non-minimal coupling

Now we will consider coupling constant $\xi > 0$. The potential shown at Fig.5 allows also for an undesirable behaviour associated with the red ball, probably leading to eternal inflation. Here we will only consider the green one. For $\phi \gg M_P/\sqrt{\xi}$:

$$
U \approx \frac{\lambda_* \phi^4}{4N_F^2\left(1+\frac{\xi\phi^2}{M_P^2}\right)^2}\left(\frac{\phi}{\mu_0}\right)^{-\frac{16}{19}\delta} \to \frac{\lambda_* M_P^4}{4N_F^2\xi^2}\left(\frac{\phi}{\mu_0}\right)^{-\frac{16}{19}\delta}.
$$

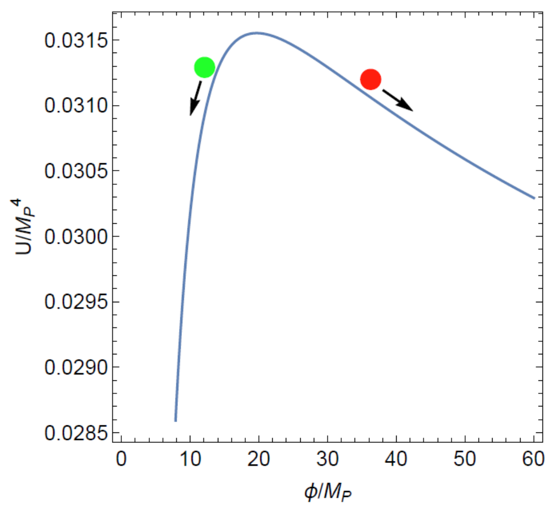

Figure 5: The non-minimally coupled potential for δ=0.1, N_F=10, ξ=1/6 [17].

Having introduced a new parameter, there is a wide range of them giving really accurate results as shown at Fig.6. It also allows us to choose $\delta << 1$. We can also calculated initial and final values of field (B.3 Fig11)and number of flavours dependent on δ and ξ (B.3 Fig.12).

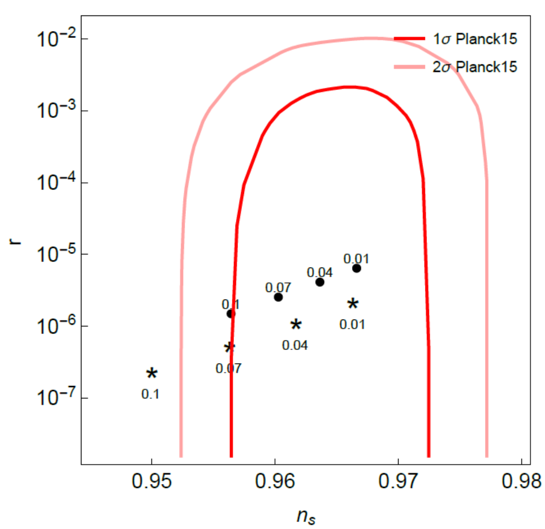

Figure 6: Full dots refer to the conformal coupling choice for ξ=1/6 and the stars to ξ=10^3, while numbers refer to value of δ [17].

6 Asymptotic safety in quantum gravity

6.1 Introduction to asymptotically safe inflation - Weinberg's idea

Asymptotically safe theories are ultraviolet complete and may be applied to physics of very short distances, in particular the early Universe. Steven Weinberg has suggested [4] that the cosmological inflation may be governed by an asymptotically safe theory.
Starting with a completely general generally covariant effective action with an ultraviolet cutoff Λ:

$$I_\Lambda[g] = -\int d^4x\sqrt{-\det g}\,\big[\Lambda^4 g_0(\Lambda) + \Lambda^2 g_1(\Lambda)\,R + g_{2a}(\Lambda)\,R^2 +$$
$$+ g_{2b}R^{\mu\nu}R_{\mu\nu} + \Lambda^{-2}g_{3a}(\Lambda)\,R^3 + \Lambda^{-2}g_{3b}(\Lambda)\,RR^{\mu\nu}R_{\mu\nu} + \ldots\big]$$

Every possible Riemann curvature tensor truncation is contained in "...". Parameters g_n are dimensionless, hence they satisfy Renormalization Group Equations:

$$\Lambda\frac{dg_n}{d\Lambda} = \beta_n(g(\Lambda)).$$

The condition for a fixed point in $g_n = g_{n_*}$ is $\beta_n(g_*(\Lambda)) = 0$ for all n. In the context of the inflation, a metric of interest is a general FRW metric. Thanks to the symmetries of the FRW metric the classical 10 Einstein equations may be reduced to one differential equation:

$$\mathcal{N}_\Lambda = \mathcal{I}_\Lambda - H\frac{\partial\mathcal{I}}{\partial H} + (-\dot{H} + 3H^2)\frac{\mathcal{I}_\Lambda}{\partial\dot{H}} + H\frac{d}{dt}\left(\frac{\mathcal{I}_\Lambda}{\partial\dot{H}}\right) + \cdots = 0.$$

In case of the inflation the Hubble parameter is constant and $\mathcal{N}_\Lambda$ simplifies. The equation for $H(t) = \bar{H}$:

$$0 = -g_0(\Lambda) + 6g_1(\Lambda)(\bar{H}/\Lambda)^2 - 864g_{3a}(\Lambda)(\bar{H}/\Lambda)^6 - 216g_{3b}(\Lambda)(\bar{H}/\Lambda)^6 + \ldots.$$

We may find the solutions at a scale at which the couplings approach their fixed points. We choose such a cut-off of Λ, that the radiative corrections are minimized $\Lambda \sim \bar{H}$. The classical solutions to the de Sitter Universe describe Eternal Inflation. A more realistic model would remain close to the de Sitter solution and after time $1/\bar{H}$ it would gradually end the inflation. By considering the perturbations of the Hubble parameter around $\bar{H}$:

$$H(t) = \bar{H} + \delta H$$

one finds, that the equations of motion indeed predict the exit from the inflationary regime. The perturbations δH are of the form:

$$\delta H \sim \exp\big\{\big(\xi\bar{H}t\big)\big\}.$$

For the positive real part of ξ instability increases, ending inflation after time $1/\xi$. This result shows a pathway that may be followed in constructing the inflationary model from the general asymptotically safe theory of gravitation.

6.2 Platania's realisation

Asymptotic safety is a theory based on the existence of a non-gaussian fixed point in the renormalization group in the ultraviolet limit. Renormalization provides us with a scale-invariant, dimensionless, finite coupling constant in high energies. Asymptotically safe models allow to avoid the primordial singularity as the gravitational interactions weaken in high energies. The Planck mission, of which the goal is the research on anisotropies in the cosmic microwave background, is supposed to gather data needed to constrain models of the early Universe, including the cosmological constant and Newton's constant [1].

The inflation occured in planckian scales, therefore under high energies and densities conditions. In this scale the effective action is dependent on the non-gaussian fixed point. In the Einstein-Hilbert truncation running Λ_k and g_k:

$$\begin{cases} g_k = g_* c_1^1 (\frac{1}{M_{Pl}})^{-\theta_1} + c_2 e_1^2 (\frac{k}{M_{Pl}})^{-\theta_2} \\ \lambda_k = \lambda_* c_1^1 (\frac{1}{M_{Pl}})^{-\theta_1} + c_2 e_1^2 (\frac{k}{M_{Pl}})^{-\theta_2} \end{cases} \tag{2}$$

are the parameters of the action, given by [11]:

$$S_{EH} = \int d^4x \sqrt{-g} \frac{1}{16\pi g_k} (R - 2\Lambda_k). \tag{3}$$

In the fixed point the scale setting condition [11] is:

$$k^2 = \frac{R}{4\Lambda^*}. \tag{4}$$

The relationship between running constants and the fixed point [11] is given by:

$$\begin{cases} g_k = \frac{g^*}{k^2} \\ \Lambda_k = \Lambda^* k^2. \end{cases} \tag{5}$$

Thus, the action in the fixed point can be described as:

$$S^*_{grav} = \int d^4x \sqrt{-g} \frac{R^2}{128\pi g_* \lambda_*}. \tag{6}$$

θ_i are the critical exponents, containing the way the renormalization group trajectories emerge from the non-gaussian fixed point [10].

The slow roll inflation conditions depend on the scalar field potential energy $V(\phi)$ and are given by:

$$\begin{cases} \epsilon(\phi) = \frac{1}{2\kappa}(\frac{V(\phi)}{V'(\phi)}) << 1 \\ \eta(\phi) = \frac{1}{\kappa}(\frac{V''(\phi)}{V(\phi)}) << 1. \end{cases} \tag{7}$$

The dominance of the scalar field's potential determines inflation. Violation of this condition ($\epsilon = 1$) results at the end of this period. If the violation is never satisfied, the eternal inflation occurs.

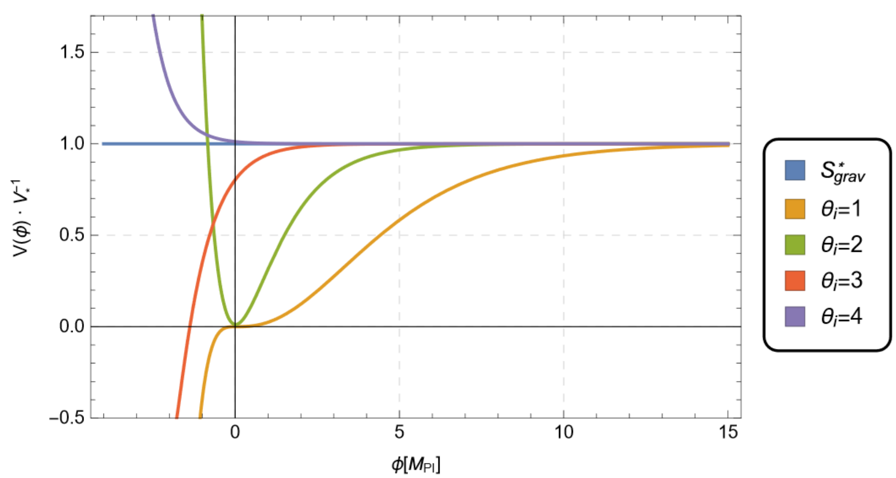

Figure 7: Slow roll inflation scalar field potential for various critical exponents θ_i, which determine the dynamics of exponential inflation.

In the Einstein frame the scalar potential is given by:

$$V(\phi) = V_* + \delta V(\phi), \tag{8}$$

where $V(\phi)$ is a variation of the scalar potential and $V_*(\phi)$ is the non-gaussian fixed point potential:

$$V_*(\phi) = 8\pi g_* \lambda_* M_{Pl}^4. \tag{9}$$

The mass scale is dependent on the non-gaussian fixed point potential:

$$\frac{m^2}{M_{Pl}^2} = \frac{4}{3} V_*(\phi). \tag{10}$$

Coupling the gravitaional action with a scalar field ϕ in the Einstein frame [12] results in obtaining:

$$S_{grav} = S^*_{grav} = \int d^4x\sqrt{-g_E}(\frac{R_E}{16\pi G_0} + \frac{1}{2}g_E^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)) \tag{11}$$

The family of scalar potential function is shown on the Figure 8. On the upper chart there is shown the dynamics in the $V_-(\phi)$, on the lower one - $V_+(\phi)$.

$$\begin{aligned} V_\pm = & \frac{m^2 e^{-2\sqrt{\frac{2\kappa}{3}\phi}}}{256\kappa}\Big\{192(e^{\sqrt{\frac{2\kappa}{3}\phi}} - 1)^2 - 3\alpha^4 + 128\Lambda - \\ & - \sqrt{32}\alpha[(\alpha^2 + 8e^{\sqrt{\frac{2\kappa}{3}\phi}} - 8) \pm \alpha\sqrt{\alpha^2 + 16e^{\sqrt{\frac{2\kappa}{3}\phi}} - 16}]^{\frac{3}{2}} - \\ & - 3\alpha^2(\alpha^2 + 16e^{\sqrt{\frac{2\kappa}{3}\phi}} - 16) \mp 6\alpha^3\sqrt{\alpha^2 + 16e^{\sqrt{\frac{2\kappa}{3}\phi}} - 16}\Big\} \end{aligned} \tag{12}$$

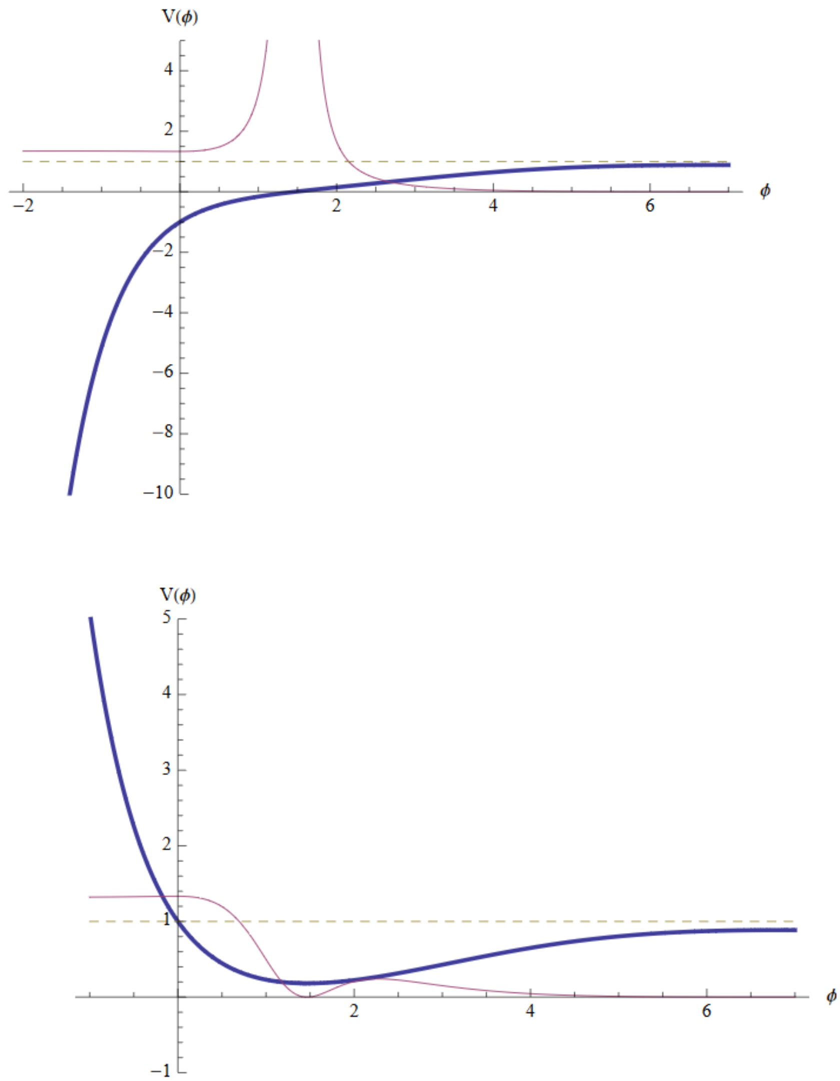

Figure 8: Scalar field potential in the Einstein frame.

7 Conclusions

The idea behind AS in quantum gravity is to describe a fundamental theory of quantum gravity by invoking the Wilsonian renormalization group, and construct its phenomenology by considering the theories emanating from the NGFP.

Results presented in section 5 show, that indeed inflation driven by AS particle theory is a viable scenario. A more complicated, non-minimal case gives us a better fit and greater freedom, but still, both ways are possible. Both lead to remarkably high numbers of flavors and colors (Fig.10, Fig.12), way beyond current knowledge. In this model, gravity has a somehow limited role and one should think about more complex settings. For example, considering asymptotically safe gravity, it might be possible to get a unified interacting fixed point of gravity and matter. We should also investigate the "red ball" behavior in the non-minimal coupled case. It is also possible to perform calculations in other regimes.

In paragraph 6.2. we have shown that the inflation's process is determined by the scalar field's potential. The dynamics of the expansion depends on the critical exponents, which describe the way of emergence of the renormalization group trajectories from the fixed point. Violation of the dominance of the scalar field's potential condition leads to the end of inflation. In some of the models this violation is never satisfied, resulting in the eternal inflation.

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Appendices

A Jordan and Einstein Frame

A.1 One Field Case

We will work in D space-time dimensions our metric has signature $(-,+,+,+,+,...)$. We take the Christoffel symbols to be

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}[\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}], \tag{13}$$

and the Riemann tensor to be

$$R^{\lambda}_{\mu\nu\sigma} = \partial_\nu \Gamma^{\lambda}_{\mu\sigma} - \partial_\sigma \Gamma^{\lambda}_{\mu\nu} + \Gamma^{\eta}_{\mu\sigma}\Gamma^{\lambda}_{\eta\nu} - \Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\eta\sigma} \tag{14}$$

The Ricci tensor and Ricci curvature scalar follow upon contractions of the Riemann tensor:

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu} \tag{15}$$

$$R = g^{\mu\nu}R_{\mu\nu} \tag{16}$$

In the single-field case the action is given by

$$S = \int d^D x\sqrt{-g}\left[f(\phi)R - \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)\right] \tag{17}$$

We will assume that $f(\phi)$ is positive definite. The frame in which $f(\phi) \neq$ constant appears in the action, as in Eq. (5), is often referred to as the Jordan frame. We will assume natural units ($c = \hbar = 1$) and take the metric tensor, $g_{\mu\nu}$, to be dimensionless. We may further parametrise

$$M_{(D)}^{D-2} \equiv \frac{1}{8\pi G_D} \tag{18}$$

in terms of a (reduced) Planck mass in D dimensions. We may make a conformal transformation of the metric, defined as

$$\hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \tag{19}$$

We assume that $\Omega(x)$ is real and therefore that $\Omega^2(x)$ is positive definite. Note that we have not made a coordinate transformation; the coordinates x^μ remain the same in each frame. From equation (7), we immediately see that

$$\hat{g}^{\mu\nu} = \frac{1}{\Omega^2(x)} g^{\mu\nu} \tag{20}$$

$$\sqrt{-\hat{g}} = \Omega^D(x)\sqrt{-g} \tag{21}$$

Upon making the transformation of equations (8)-(9), one may compute the Christoffel symbols and the Ricci curvature scalar in the new frame. One finds

$$\hat{\Gamma}^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} + \frac{1}{\Omega}\left[\Delta^\alpha_\beta \nabla_\gamma \Omega + \Delta^\alpha_\gamma \nabla_\beta \Omega - g_{\beta\gamma}\nabla^\alpha \Omega\right] \tag{22}$$

$$\hat{R} = \frac{1}{\Omega^2}\left[R - \frac{2(D-1)}{\Omega}\Box\Omega - (D-1)(D-4)\frac{1}{\Omega^2} g^{\mu\nu}\nabla_\mu \Omega \nabla_\nu \Omega\right], \tag{23}$$

where

$$\Box\Omega = g^{\mu\nu}\nabla_\mu\nabla_\nu\Omega = \frac{1}{\sqrt{-g}}\partial_\mu[\sqrt{-g}g^{\mu\nu}\partial_\nu\Omega] \tag{24}$$

One must be careful to specify whether one is taking derivatives with respect to the original metric, $g_{\mu\nu}$, or the transformed metric, $\hat{g}_{\mu\nu}$ because the Christoffel symbols transform in Ω-dependent ways under the transformation of equations (8)-(9). Using Eqs. (7) - (12), we may rewrite the first term in the action, involving R:

$$\int d^D x\sqrt{-g}f(\phi)R = \int d^D x\frac{\sqrt{-\hat{g}}}{\Omega^D}f(\phi)\left[\Omega^2\hat{R} + \frac{2(D-1)}{\Omega}\Box\Omega - \right.$$
$$\left. (D-1)(D-4)\frac{1}{\Omega^2}g^{\mu\nu}\nabla_\mu\Omega\nabla_\nu\Omega\right] \tag{25}$$

Let us look at each of these terms in turn. The first term on the right hand side becomes

$$\int d^D x\sqrt{-\hat{g}}\left[\left(\frac{f}{\Omega^{D-2}}\right)\hat{R}\right] \tag{26}$$

To obtain the canonical Einstein-Hilbert gravitational action in the transformed frame, we identify

$$\Omega^{D-2}(x) = \frac{2}{M_{(D)}^{D-2}} f\left[\phi(x)\right] \tag{27}$$

We may integrate the second term on the right hand side of equation (13) by parts. Note that the operator $\Box$ acting on Ω is defined in terms of the original metric, $g_{\mu\nu}$, rather than the transformed metric. Using Eqs. (8),(9), (12) and (15), we find

$$\begin{aligned}\int d^D x\sqrt{-\hat{g}}\frac{2(D-1)}{\Omega^{D+1}} f\Box\Omega = \\ \int d^D x\sqrt{-\hat{g}}(D-1)(D-3)M_{(D)}^{D-2}\frac{1}{\Omega^2}\hat{g}^{\mu\nu}\hat{\nabla}_\mu\Omega\hat{\nabla}_\nu\Omega\end{aligned} \tag{28}$$

Recall that x^μ is unaffected by the conformal transformation, so that $\hat{\partial}_\mu = \partial_\mu$. Because the covariant derivatives in equation (16) act only on scalar functions, we have $\nabla_\mu\Omega = \partial_\mu\Omega$, and hence $\hat{\nabla}_\mu\Omega = \nabla_\mu\Omega$. The last term on the right hand side of equation (13) is

$$\begin{aligned}\int d^D x\sqrt{-\hat{g}}(D-1)(D-4)\left(\frac{f}{\Omega^{D+2}}\right) g^{\mu\nu}\nabla_\mu\Omega\nabla_\nu\Omega = \\ \int d^D x\sqrt{-\hat{g}}\frac{1}{2}(D-1)(D-4)M_{(D)}^{D-2}\frac{1}{\Omega^2}\hat{g}^{\mu\nu}\hat{\nabla}_\mu\Omega\hat{\nabla}_\nu\Omega\end{aligned} \tag{29}$$

Combining Eqs. (12), (15), and (16) we find

$$\begin{aligned}\int d^D x\sqrt{-g} f(\phi) R = \int d^D x\sqrt{-\hat{g}}\frac{M^{D-2}}{2} f(\phi)\Big[\hat{R} - \\ (D-1)(D-4)\frac{1}{\Omega^2}\hat{g}^{\mu\nu}\hat{\nabla}_\mu\Omega\hat{\nabla}_\nu\Omega\Big]\end{aligned} \tag{30}$$

The gravitational portion of the action now includes a canonical Einstein-Hilbert term. For this reason, the frame corresponding to $\hat{g_{\mu\nu}}$ is often referred to as the Einstein frame.
We may next consider how the scalar field's kinetic and potential terms in the action transform under $g_{\mu\nu} \to \hat{g}_{\mu\nu}$. We have

$$\begin{aligned}\int d^D x\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)\right] = \\ \int d^D x\sqrt{-\hat{g}}\left[-\frac{1}{2}\frac{1}{\Omega^{D-2}}\hat{g}^{\mu\nu}\hat{\nabla}_\mu\phi\hat{\nabla}_\nu\phi - \hat{V}\right]\end{aligned} \tag{31}$$

where we have introduced a transformed potential,

$$\hat{V} \equiv \frac{V}{\Omega^D} \tag{32}$$

The full action of Eq. (5) may then be written

$$\int d^D x \sqrt{-\hat{g}} \left[\frac{M_{(D)}^{D-2}}{2} \hat{R} - \frac{1}{2}(D-1)(D-2) M_{(D)}^{D-2} \frac{1}{\Omega^2} \hat{g}^{\mu\nu} \hat{\nabla}_\mu \Omega \hat{\nabla}_\nu \Omega \right.$$
$$\left. - \frac{1}{2} \frac{1}{\Omega^{D-2}} \hat{g}^{\mu\nu} \hat{\nabla}_\mu \phi \hat{\nabla}_\nu \phi - \hat{V} \right] \tag{33}$$

Upon substituting f for Ω using Eq. (15), the action in the transformed frame becomes

$$\int d^D x \sqrt{-g} \left[\frac{M_{(D)}^{D-2}}{2} \hat{R} - \frac{(D-1)}{(D-2)} M_{(D)}^{D-2} \frac{1}{f^2} \hat{g}^{\mu\nu} \hat{\nabla}_\mu f \hat{\nabla}_\nu f - \right.$$
$$\left. \frac{1}{4f} M_{(D)}^{D-2} \hat{g}^{\mu\nu} \hat{\nabla}_\mu \phi \hat{\nabla}_\nu \phi - \hat{V} \right] \tag{34}$$

In the single-field case, we may next rescale the field, $\phi \to \phi$ so that the new scalar field in the transformed frame has the canonical kinetic term in the action of Eq. (22). We define $\hat{\phi}$ such that

$$-\frac{1}{2} \hat{g}^{\mu\nu} \hat{\nabla}_\mu \hat{\phi} \hat{\nabla}_\nu \hat{\phi} = -\frac{M_{(D)}^{D-2}}{4f} \hat{g}^{\mu\nu} \left[\hat{\nabla}_\mu \phi \hat{\nabla}_\nu \phi + \frac{2(D-1)}{(D-2)} \frac{1}{f} \hat{\nabla}_\mu f \hat{\nabla}_\nu f \right] \tag{35}$$

We assume

$$\frac{d\hat{\phi}}{d\phi} = F(\phi) \tag{36}$$

in terms of some as-yet unspecified function F. In the single-field case, we also have $f = f(\phi)$, so that equation (22) yields

$$F(\phi) = \left(\frac{d\hat{\phi}}{d\phi} \right) = \sqrt{\frac{M_{(D)}^{D-2}}{2f^2(\phi)}} \sqrt{f(\phi) \frac{2(D-1)}{(D-2)} [f'(\phi)]^2} \tag{37}$$

where primes denote derivatives with respect to ϕ. In terms of the rescaled field, the action of equation (22) may be written

$$\int d^D x \sqrt{-g} \left[f(\phi) R - \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) \right] =$$
$$\int d^D x \sqrt{-\hat{g}} \left[\frac{M_{(D)}^{D-2}}{2} \hat{R} - \frac{1}{2} \hat{g}^{\mu\nu} \hat{\nabla}_\mu \hat{\phi} \hat{\nabla}_\nu \hat{\phi} - \hat{V}(\hat{\phi}) \right] \tag{38}$$

The action in the second line now has both the canonical Einstein-Hilbert form for the gravitational portion as well as the canonical kinetic term for the scalar field.

A.2 Two field analysis

Let ϕ^1 ϕ^2 be scalar fields coupled to gravity, for D dimensions we have the following action:

$$\int d^D x\sqrt{|g|}\left[-f(\phi^1,\phi^2)R+\frac{1}{2}\delta^i_j g^{\mu\nu}\nabla_\mu\phi^i\nabla_\nu\phi^j-V(\phi^1,\phi^2)\right] \tag{39}$$

The steps of the transformation for the gravitational part are exactly the same and gives,

$$\int d^D x\sqrt{|g|}f(\phi^i)R \;=\; \int d^D x\sqrt{|g|}\left[\frac{1}{2}\hat{R}-\frac{1}{2}\frac{(D-1)}{(D-2)}\frac{1}{f^2}\hat{g}^{\mu\nu}\hat{\nabla}_\mu f\hat{\nabla}_\nu f\right], \tag{40}$$

where

$$\hat{\nabla}_\mu f=(\hat{\nabla}_\mu\phi^i)f_{,i} \tag{41}$$

The scalar part transforms similarly:

$$\int d^D x\sqrt{|g|}\left[\frac{1}{2}\delta^i_j g^{\mu\nu}\nabla_\mu\phi^i\nabla_\nu\phi^j-V(\phi^1,\phi^2)\right]= \\ \int d^D x\sqrt{|\hat{g}|}\left[\frac{1}{4f}\delta_{ij}\hat{g}^{\mu\nu}\hat{\nabla}_\mu\phi^i\hat{\nabla}_\nu\phi^j-\hat{V}\right] \tag{42}$$

So finally the action becomes:

$$\int d^D x\sqrt{|\hat{g}|}\left[-\frac{1}{2}\hat{R}+\frac{(D-1)}{(D-2)}\frac{1}{f^2}\hat{g}^{\mu\nu}\hat{\nabla}_\mu f\hat{\nabla}_\nu f \right. \\ \left. -\frac{1}{4f}\delta^i_j\hat{g}^{\mu\nu}\hat{\nabla}_\mu\phi^i\hat{\nabla}_\nu\phi^j-\hat{V}\right] \tag{43}$$

There arises a question whether there is such a field transformation that gives canonical kinetic term structure. Let us rewrite the action in terms of metric in field space $\mathcal{G}_{ij}$.

$$\int d^D x\sqrt{|g|}\left[-R+\frac{1}{2}\mathcal{G}_{ij}jg^{\mu\nu}\nabla_\mu\phi^i\nabla_\nu\phi^j-V(\phi^1,\phi^2)\right], \tag{44}$$

with

$$\mathcal{G}_{ij}=\frac{1}{2f}\delta_{ij}+\frac{(D-1)}{(D-2)}\frac{1}{f^2}f_{,i}f_{,j} \tag{45}$$

The necessary condition for the conformal transformation: $\mathcal{G}_{ij}\to\hat{\mathcal{G}}_{ij}=\delta_{ij}$ to exist is that all the Riemann tensors coefficients vanish. Let us first rescale $\mathcal{G}_{ij}$:

$$\hat{\mathcal{G}_{\rangle|}}=2f\mathcal{G}_{ij}=\delta_{ij}+\frac{2(D-1)}{(D-2)}\frac{1}{f}f_{,i}f_{,j} \tag{46}$$

then

$$\hat{R}$$

takes the following form:

$$\hat{R} = \frac{2(D-1)(D-2)}{L(\phi)}$$

$$(2ff_{11}f_{22} - f_{,1}^2 f_{,22} - f_{,2}^2 f_{,11} - 2f_{,12}(ff_{,12} - f_{,1}f_{,2})), \quad (47)$$

where

$$L(\phi^i) = \left[(D-2)f + 2(D-1)\sum_i f_{,i}^2\right] \quad (48)$$

We denote $\phi^1 = \phi$ and $\phi^2 = \chi$ and take $f(\phi, \chi)$ as:

$$f(\phi, \chi) = \frac{1}{2}[M^{D-2} + \xi_\phi \phi^2 + \xi_\chi \chi^2] \quad (49)$$

We obtain:

$$L(\phi, \chi)\hat{R} = 2(D-1)(D-2)\xi_\phi \xi_\chi M^{D-2} \quad (50)$$

therefore we see that, for $D > 2$ only if $M = 0$ one can find such a conformal transformation that would bring both the gravitational and kinetic terms into canonical form. This implies a further condition on ξ_i, namely:

$$\frac{\xi_\phi \phi^2 + \xi_\chi \chi^2}{M^{D-2}} \gg 1 \quad (51)$$

to obtain the canonical Einstein and kinetic term.

B Asymptotically Safe Theory

B.1 Potential of Asymptotically Safe Theories

Let us recall that AS theory can be formulated as:

$$\begin{aligned} Rep\,(SU(N_C)) &= \{H\} \subset M(N_F \times N_F, \mathbb{C}), \\ \delta &= \frac{N_F}{N_C} - \frac{11}{2}, \\ H_{ii} &= \phi/\sqrt{2N_F}, \end{aligned}$$

where we consider such a limit of N_F - number of flavours and N_C - number of colours, that $\delta > 0$ is a constant parameter and ϕ is our field along the diagonal of H. Potential for an AS theory is given by [16]:

$$\begin{aligned} V_{\text{iUVFP}} &= \frac{\lambda_* \phi^4}{4N_F^2\,(1 + W(\phi))}\left(\frac{W(\phi)}{W(\mu_0)}\right), \\ \lambda_* &= \delta \frac{16\pi^2}{19}\left(\sqrt{20 + 6\sqrt{23}} - \sqrt{23} - 1\right), \end{aligned}$$

where $W(\phi) := W(z(\phi))$ is a Lambert function, the solution of equation below:

$$\begin{aligned}
z(\mu) &= W exp W, \\
z(\mu) &= \left(\frac{\mu_0}{\mu}\right)^{\frac{4}{3}\delta\alpha^*}\left(\frac{\alpha^*}{\alpha(\mu_0)}-1\right) exp\left(\frac{\alpha^*}{\alpha(\mu)-1}\right), \\
\alpha^* &= \frac{26}{57}\delta + \mathcal{O}(\delta^2), \\
\alpha(\mu) &= \frac{\alpha^*}{1+W(\mu)},
\end{aligned}$$

where α^* is a gauge coupling at UV fixed point and $\alpha(\mu_0)$ is the same coupling at a reference scale μ_0. Now we may consider a renormalisation with $k \in \mathbb{R}_+$, eg. $k = \frac{1}{2}$:

$$\begin{aligned}
\alpha_0 &= \frac{\alpha^*}{1+k} = \frac{2\alpha^*}{3}, \\
\alpha(\mu) &= \alpha^* + (\alpha(\mu_0) - \alpha^*)\left(\frac{\mu}{\mu_0}\right)^{-\frac{104}{171}\delta^2 + \mathcal{O}(\delta^3)}, \\
\lim_{\phi/\mu_0 \to \infty} W(\phi) &= k\left(\frac{\phi}{\mu_0}\right)^{-\frac{104}{171}\delta^2}.
\end{aligned}$$

Finally:

$$\lim_{\phi/\mu_0 \to \infty} V_{\text{iUVFP}} = \frac{\lambda_* \phi^4}{4N_F^2}\left(\frac{\phi}{\mu_0}\right)^{-\frac{16}{19}\delta}.$$

B.2 Additional plots for minimally coupled case

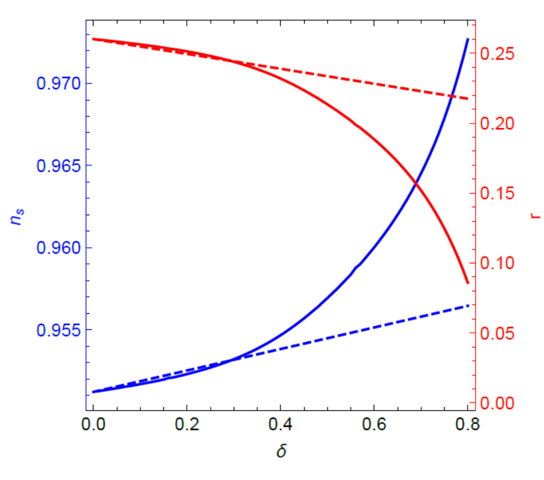

Figure 9: The solid lines are calculated using the complete expression for the potential. The dashed lines show the leading order in δ [17].

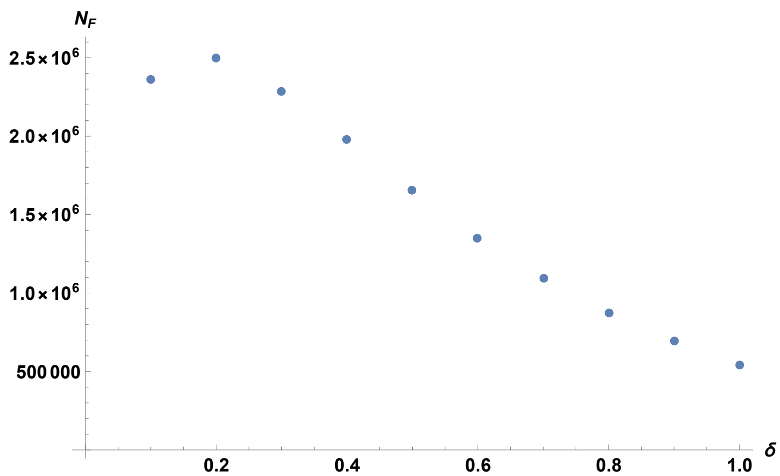

Figure 10: N_F as a function of δ based on amplitudes of scalar perturbations measured by Planck. For δ=0.8, $N_F = 873009$, $N_C = 138573$.

B.3 Additional plots for non-minimally coupled case

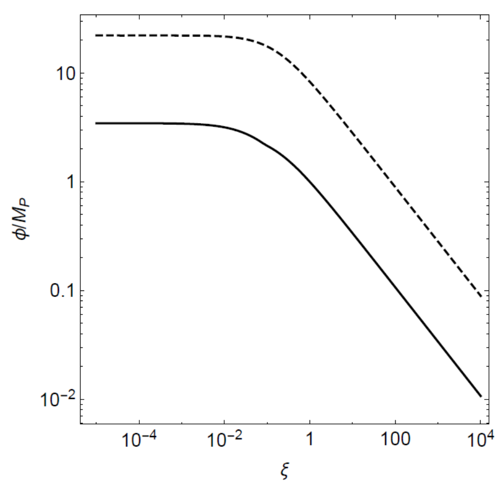

Figure 11: Initial (dashed-line) and final (solid-line) values of the inflaton field in the Jordan frame as function of the non-minimal coupling for $\delta = 0.01$ [17][18].

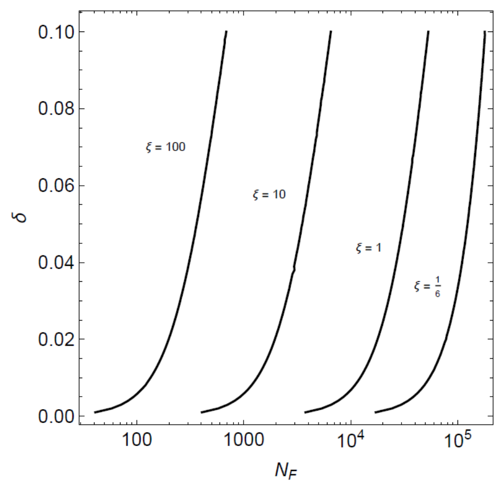

Figure 12: δ-dependence on N_F for different values of the non-minimal coupling obtained by constraining the model to provide the observed amplitude of density perturbations [17].